Discrete Mathematics: Combinatorics and Graph Theory

Exam 3

Instructions. Solve any 5 questions and state which 5 you would like graded. Write neatly and show your work to receive full credit. You must sign the attendance sheet when returning your booklet. Good luck!

1. Solve the following recurrence relations.
   (a) $a_n = 4a_{n-1} - 4a_{n-2}$ for $n \geq 2, a_0 = 6, a_1 = 8$
   (b) $a_n = 2a_{n-1} + 5a_{n-2} - 6a_{n-3}$ with $a_0 = 7$, $a_1 = -4$, and $a_2 = 8$.
   (c) $a_n = na_{n-1}$, $a_1 = 1$
   (d) $a_n = a_{n-1} + 2n$ with $a_0 = 2$
   (e) $a_{n+2} + a_{n+1} - 6a_n = 2^n$ for $n \geq 0$.
2. Let E and F be the events that a family of n children has children of both sexes and has at most one boy, respectively. Are E and F independent if
   (a) $n = 2$.
   (b) $n = 4$.
   (c) $n = 5$.
3. Consider a password generated by selecting characters from a three letter alphabet α, β or γ which must use each letter at least once. How many such passwords of length 8 are there?
4. Suppose that there are two slot machines, one of which pays out 10% of the time and the other pays out 20% of the time. Unfortunately, you have no idea which is which. Suppose you randomly choose a machine and put in a quarter. If you don't get a jackpot, what is the chance that you chose the machine that pays out 20% of the time? If you had instead gotten a jackpot, what would be the chance that you chose the one that pays out 20% of the time?
5. Derive a formula for the k^{th} factorial moment of the Poisson distribution. Evaluate the expression

$$\mathbb{E}[X(X-1)(X-2)\cdots(X-k+1)]$$

   Hint: write the expression as a summation and simplify completely.
6. Define X_1 and X_2 as independent Poisson random variables with parameters λ_1 and λ_2. Let $\lambda = \lambda_1 + \lambda_2$ and define $Z = X_1 + X_2$. Find the distribution of Z by evaluating the expression $P(Z = z)$.
7. Let A_n be the $n \times n$ matrix with 2's on its main diagonal, 1's in all positions next to a diagonal element, and 0's everywhere else. Find a recurrence relation for d_n, the determinant of A_n. Solve this recurrence relation to find a formula for d_n.